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ON EXISTENCE AND UNIQUENESS OF GENERATORS OF SHY SETS IN POLISH GROUPS

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Abstract

By using a technique of invariant measures developed by Oxtoby [11] for a Polish group, which is dense in itself, we prove an existence of a quasi-finite left (or right) invariant generator of left (or right)-shy sets in an entire group and establish that no any element of the class of all quasi-finite generators of left (or right)-shy sets possesses a uniqueness property. We get the validity of an analogous result for quasi-finite generators of two-sided-shy sets in Polish groups, which are dense in itself and are equipped with two-sided invariant metrics. These results allows us to answer to Questions 2.1-2.2 posed in [17].

1. Introduction

In [15], for a Polish topological vector space $\mathbb{V}$ has been introduced the notion of a generator of shy sets μ, which is such a Borel measure in $\mathbb{V}$ that a condition $\overline{\mu}(X) = 0$ implies that X is Haar null (or shy) in the sense [6], where $\overline{\mu}$ denotes a usual completion of the μ. Here has been shown that the class of generators of shy sets in $\mathbb{V}$ contains specific Borel

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measures, which generate implicitly introduced subclasses of shy sets (see, for example, [2], [3], [6], [8], [20], [22], [23]). Moreover, such measures possess many interesting properties (see, for example, [12], [15]). Some applications of generators of shy sets in infinite-dimensional analysis can be found in [13-15].

In [17], it has been shown that the class of generators of shy sets in an abelian Polish group G is non-empty, if G contains any uncountable locally compact Hausdorff topological subgroup. This result (see [17], Theorem 2.1) extends a certain result early obtained for Polish topological vector spaces in [15] (see Theorem 2.1, p. 238). For a Borel probability measure μ in a Polish topological vector space $\mathbb{V}$, a generator of shy sets $\mathbf{G}_\mu$ has been constructed such that a subclass of shy sets generated by the measure μ coincides with the class of $\overline{\mathbf{G}}_\mu$-zero sets, where $\overline{\mathbf{G}}_\mu$ denotes a usual completion of the generator $\mathbf{G}_\mu$ (see [15], Theorem 3.1, p. 245).

A notion of Haar measure zero (or shy) in abelian Polish groups to all non-abelian Polish groups has been given in Mycielski [9]. Like this approach, the notion of generators of shy sets introduced for Polish topological vector spaces [15] has been extended to all Polish groups in [17], where an example of a two-sided invariant generator of two-sided-shy sets has been constructed in the product of unimodular Polish groups and Dougherty's criterion of shyness has been extended. In the same paper have been posed the following unsolving problems:

Problem 1.1 ([17], Question 2.1, p. 5)**.** Let G be a Polish group. Whether there exists a generator of two-sided (left or right)-shy sets in G?

Problem 1.2 ([17], Question 2.1, p. 5)**.** Let G be a Polish group and let a class of generators of two-sided (left or right)-shy sets in G is non-empty. Whether there exists a generator of two-sided (left or right)-shy sets with the property of uniqueness in the entire class of generators?

So each a quasi-finite two-sided (left or right) invariant Borel measure in Polish group same times is a generator of two-sided (left or right)-shy sets in the entire group, we show that a solution of Problems 1.1-1.2 is directly connected with the problem of a measure in groups, which is the problem of defining a two-sided (left or right) invariant Borel measure, with specified properties, in a given group. In this direction can be mentioned especially the paper [11], which contains a solution of the problem of an existence and uniqueness of a quasi-finite left-invariant Borel measure in a complete separable metric group, which is dense in itself. In this paper has been shown that there exists a quasi-finite left-invariant Borel measure in any Polish group, which is dense in itself, but a locally finite Borel measure is possible only if that group is locally compact. Take into account the validity of these facts, Oxtoby stated Haar's theorem in the following generalized form: *In a complete separable metric group*, *there exists a locally finite left-invariant Borel measure*, *if and only if the group is locally compact and dense in itself*.

The generalized form of Haar's theorem stated above is an immediate consequence of the following theorem, the proof of which is due to Oxtoby and Ulam [10] (see p. 561, footnote 3).

Proposition 1.1. *Let G be any complete separable metric group*, *which is not locally compact*, *and let m be any left-invariant Borel measure in G. Then every neighbourhood contains non-denumerably many disjoint mutually congruent sets of equal finite positive measure*.

By a refinement of the foregoing reasoning, Oxtoby has proved the following stronger theorem:

Proposition 1.2 ([11], Theorem 2, p. 217)**.** *Let G be any complete separable metric group*, *which is not locally compact*, *and let m be any left-invariant Borel measure in G. Then any neighbourhood contains a compact set*, *which is the union of non-denumerably many disjoint mutually congruent compact sets of equal finite positive measure*.

Remark 1.1. Concerning with Proposition 1.2, it can be noted that there exists a complete separable metric group G, which is locally compact and a left-invariant non-locally finite Borel measure μ in the entire group such that every neighbourhood contains a compact set, which is the union of non-denumerably many disjoint mutually congruent compact sets of equal finite positive measure. Indeed, in order to give such an example, it is sufficient to set $G = R^2$ (with a usual Euclidian topology) and to consider under μ a one-dimensional Preiss-Tiser generator of shy sets in G (see [15], Section 7, p. 255).

The purpose of the present paper is to answer to Problems 1.1-1.2 by a technique of invariant measure theory developed by Oxtoby in [11].

The paper is organized as follows:

Section 2 contains some auxiliary notions and propositions from the theory of invariant measures in Polish groups. In Section 3, we give solutions to Problems 1.1-1.2 for generators of left (or right)-shy sets in a Polish group, which is dense in itself. In addition, we give solutions to the same problems for generators of two-sided-shy sets in a Polish group, which is dense in itself and is equipped with two-sided invariant metric.

2. Some Auxiliary Propositions from the Theory of Invariant Measures in Polish Groups

Let $\mathbf{G}$ be a Polish group, by which we mean a separable group with a complete metric for which the transformation (from $\mathbf{G} \times \mathbf{G}$ onto $\mathbf{G}$), which sends $(\mathbf{x}, \mathbf{y})$ into $\mathbf{x}^{-1}\mathbf{y}$ is continuous. Let $\mathcal{B}(\mathbf{G})$ denotes the σ-algebra of Borel subsets of $\mathbf{G}$.

Definition 2.1. A Borel set $X \subseteq G$ is called *two-sided-shy*, if there exists a Borel probability measure μ over G such that $\mu(fXg) = 0$ for all $f, g \in G$. A subset of a Borel two-sided-shy set is called also *two-sided-shy*.

Definition 2.2. A Borel set $X \subseteq G$ is called *left* (or *right*)-*shy*, if there exists a Borel probability measure μ over G such that $\mu(fX) = 0$ (or $\mu(Xf) = 0$) for all $f \in G$. A subset of a Borel left (or right)-shy set is called also *left* (or *right*)-*shy*.

Definition 2.3. A Borel measure μ in G is called a *generator of two-sided-shy sets* in G, if

$$(\forall X)(\overline{\mu}(X) = 0 \to X \in S(G)),$$

where $\overline{\mu}$ denotes a usual completion of the Borel measure μ and $S(G)$ denotes a class of all two-sided-shy sets.

Definition 2.4. A Borel measure μ in G is called a *generator of left* (or *right*)-*shy sets* in G, if

$$(\forall X)(\overline{\mu}(X) = 0 \to X \in \mathcal{LS}(G) \text{ (or } \mathcal{RS}(G)),$$

where $\mathcal{LS}(G)$ and $\mathcal{RS}(G)$ denote classes of all left-shy and right-shy sets in G, respectively.

Definition 2.5. A Borel measure μ in G is called *quasi-finite*, if there exists a compact set $U \subseteq G$ for which $0 < \mu(U) < \infty$.

Definition 2.6. A Borel measure μ in G is called *semi-finite*, if for X with $\mu(X) > 0$, there exists a compact subset $F \subseteq X$ for which $0 < \mu(F) < \infty$.

Definition 2.7. A Borel measure μ in G is called *left invariant*, if

$$(\forall X)(\forall g)(X \in B(G) \,\&\, g \in G \to \mu(gX) = \mu(X)).$$

Definition 2.8. A Borel measure μ in G is called *right invariant*, if

$$(\forall X)(\forall g)(X \in B(G) \,\&\, g \in G \to \mu(Xg) = \mu(X)).$$

Definition 2.9. A Borel measure μ in G is called *two-sided invariant*, if

$$(\forall X)(\forall g, f)(X \in B(G) \,\&\, g, f \in G \to \mu(gXf) = \mu(X)).$$

Definition 2.10. A Borel measure μ in G is called *locally finite*, if there is a neighbourhood U of unity such that $0 < \mu(U) < +\infty$.

Definition 2.11. Let K be the class of measures in G. We say that a measure $\mu \in K$ has the property of uniqueness in the class K, if μ and λ are equivalent for every $\lambda \in K$.

Definition 2.12. Let G be equipped with a left invariant metric. Two sets A and B are said to be *congruent*, if there exists an element $a \in G$ such that $B = aA$.

Definition 2.13. Let G be equipped with a two sided invariant metric. Two sets A and B are said to be *congruent*, if there exists elements $a, b \in G$ such that $B = aAb$.

The following lemma plays a key role in our further considerations:

Lemma 2.1 ([11], Theorem 3, p. 220)**.** *In any complete separable metric group which is dense in itself, there exists a left-invariant quasi-finite Borel measure.*

Remark 2.1. In [11] has been demonstrated, that there always exist infinitely many left-invariant quasi-finite Borel measures and there is certainly no uniqueness theorem for the problem of measure. In case the group is locally compact, the construction may in some cases generate Haar's measure. In the additive group of real numbers, that construction gives such measures, which are not locally finite (see also, Remark 1.1).

In the sequel, we need the following auxiliary proposition:

Lemma 2.2. *Every two-sided* (*left or right*) *invariant quasi-finite Borel measure* μ *defined on a Polish group* G *is a generator of two-sided* (*left or right*)*-shy sets in* G.

Proof. We present the proof of Lemma 2.2 for a two-sided invariant quasi-finite Borel measure μ. One can get the validity of Lemma 2.2 similarly for left (or right) invariant quasi-finite Borel measures.

Let $\overline{\mu}(S) = 0$ for $S \subseteq G$. Since $\overline{\mu}(S) = 0$, there exists a Borel set S' for which $S \subseteq S'$ and $\mu(S') = 0$. By using a two-sided invariance of the Borel measure μ, we have

$$(\forall f,\ g)(f,\ g \in G \to \mu(fX'g) = 0).$$

Since μ is quasi-finite, there is a Borel set F with $0 < \mu(F) < +\infty$. We set

$$(\forall X)(X \in \mathcal{B}(G) \to \lambda(X) = \frac{\mu(X \cap F)}{\mu(F)}).$$

Let us show that λ is a two sided transverse to the Borel set S'. Indeed, we have

$$(\forall f,\ g)(f,\ g \in G \to \lambda(fX'g) = \frac{\mu((fX'g) \cap F)}{\mu(F)} \leq \frac{\mu(fX'g)}{\mu(F)} = 0).$$

The latter relation means that S' is a Borel two-sided-shy set. S being a subset of S' also is two-sided-shy set. This ends the proof of Lemma 2.2.

□

Here, we give Oxtoby's proof of Theorem 3 presented in [11] (see p. 200) with a small change in order to get the validity of the following assertion:

Lemma 2.3 (Oxtoby)**.** *In any complete separable metric group with two-sided invariant metric which is dense in itself, there exists a two-sided-invariant quasi-finite Borel measure.*

Proof. Recall that a metric ρ is two-sided invariant, if $\rho(zxh,\ zyh) = \rho(x,\ y)$ for all x, y, z, and h in G. Let $(U_k)_{k \in N}$ be a sequence of non-empty open sets in G, whose diameters tend to zero. Let $(W_k)_{k \in N}$ be a sequence of finite positive numbers tending to zero. Let U denote the family of sets in G congruent to one of the sets $\{U_k : k \in N\}$, i.e.,

$$U = \{bU_k a : a, b \in G \; \& \; k \in N\}.$$

Consider any set $A \subseteq G$, and any covering of A by a finite or infinite sequence of sets of the family U, say $A \subseteq \cup_{i \in I} b_i U_{n_i} a_i$. Form the sum $\sum_{i \in I} W_{n_i}$ and let $L_r(A)$ denote the lower bound of these sums for all covering sequences of sets U with diameters less than r. (Such coverings always exist, indeed A can be covered by a sequence of sets congruent to U_k, for any fixed k, since the space is separable.) It is evident that as r decreases, $L_r(A)$ will not decrease. Hence, the limit $m^*(A) = \lim_{r \to 0} L_r(A)$ exists or is equal to $+\infty$. In other words, m^* is defined as follows:

$$m^*(A) = \lim_{r \to 0} \inf \left\{ \sum_{i \in I} W_{n_i} : A \subseteq \cup_{i \in I} b_i U_{n_i} a_i \; \& \; \operatorname{diam}(U_{n_i}) \leq r \right\},$$

for all $A \subseteq G$.

From this definition, it is an easy to verify that the following conditions are satisfied:

(a1) $0 \leq m^*(A) \leq +\infty$;

(a2) $m^*(A) \leq m^*(B)$ if $A \subseteq B$;

(a3) $m^*(\cup_{n \in N} A_n) \leq \sum_{n \in N} m^*(A_n)$ for any sequence $(A_n)_{n \in N}$ of subsets of G;

(a4) $m^*(A \cup B) = m^*A + m^*B$, if A and B are separated by a positive distance;

(a5) $m^*(xAy) = m^*(A)$ for arbitrary $x, y \in G$ and $A \subseteq G$;

(a6) $m^*(A) = 0$ if A contains only one point;

(a7) $m^*(A) = \inf m^*(B)$ for all G_δ sets B, which contain A.

Condition (a7) follows from the fact that any set is contained in a G_δ set having the same outer measure, since only open coverings are employed.

It follows that m^* is a regular outer measure in the sense of Caratheodory [1] (see p. 238) (provided it is finite and positive for at least one set, see below) and therefore defines a left-invariant measure in G. In view of (a7), this measure is the complete extension of a Borel measure m. We shall say that the Borel measure m is the measure generated by the measuring system $\{U_n, W_n : n \in N\}$. Now, we need to show that it is always possible to find a measuring system $\{U_n, W_n : n \in N\}$, which generates a measure m such that $0 < m(A) < +\infty$ for at least one set. It is clear that not every measuring system will do this. For example, in the additive group of real numbers, if U_n is an interval of length $1/n$, and we take $W_n = n^{-2}$, every set will have measure zero. To secure a non-trivial measure, it is therefore necessary to choose the measuring system with some care.

The sets $C(n; i_l, \cdots, i_n)$ with primary index n will be said to be of rank n. To this end, we shall construct in G a family of sets $C(n; i_1, \cdots, i_n)$ $(i_1, \cdots, i_n = 0, 1; n = 1, 2, \cdots)$ with the following properties:

(i) $C(n; i_l, \cdots, i_n)$ is compact.

(ii) $C(n; i_l, \cdots, i_n) = C(n+1; i_l, \cdots, i_n, 0) \cup C(n+1; i_l, \cdots, i_n, 1)$.

(iii) The 2^n sets of rank n are mutually congruent.

(iv) The diameter d_n of the sets of rank n tends to zero as $n \to \infty$.

(v) Any two sets of rank n are separated by a distance greater than d_n.

Define $C = C(1; O) \cup C(1; 1)$. Then $C = \bigcup_{i_1 \cdots i_n} C(n; i_l, \cdots, i_n)$ for each n. It may be remarked that the family of sets $C(n; i_l, \cdots, i_n)$ evidently represent C as a dyadic discontinuum (see [5], p. 134), and therefore C is homeomorphic to the Cantor set.

To construct such a family of sets, note first that since G is dense in itself, we can choose a sequence of points x_n tending to the unity element e, such that

$$0 < (e, x_{n+1}) < (e, x_n)/9 \ (n = 1, 2, \cdots). \tag{2.1}$$

Define

$$a_{i_1 \cdots i_n} = x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} (i_l, \cdots, i_n = 0, 1).$$

Let $A(n; i_l, \cdots, i_n)$ denote the set of all points $a_{i_1 \cdots i_{n+p}} (p \geq 0)$, whose first n indices are equal to $i_l, i_2, \cdots, i_n$ and define $C(n; i_l, \cdots, i_n)$ as the closure of the countable set $A(n; i_l, \cdots, i_n)$. Evidently, we have

$$A(n; i_l, \cdots, i_n) = A(n+1; i_l, \cdots, i_n, 0) \cup A(n+1; i_l, \cdots, i_n, 1).$$

Properties (ii) and (iii) follow at once by taking closures. Furthermore, the diameter d_n of the sets $C(n; i_i, \cdots, i_n)$ of rank n is given by $d_n = \operatorname{diam} A(n; 0, \cdots, 0)$. We proceed to obtain an estimate of d_n.

Note that the left-invariance of the metric implies that for arbitrary elements y, y_1, and y_2 in G, we have

$$(e, y_l y_2) \leq (e, y_1) + (y_1, y_1 y_2) = (e, y_1) + (e, y_2).$$

Also,

$$(e, y) = (y^{-1}, e) = (e, y^{-1}),$$

so that $(e, y^k) \leq (e, y)$ if $k = -1, 0, 1$. A simple induction then shows that

$$(e, y_1^{k_1} \cdots y_n^{k_n}) < (e, y_1) + (e, y_2) + \cdots + (e, y_n),$$

for arbitrary $y_1, \cdots, y_n$ in G and exponents $k_1, \cdots, k_n$ equal to -1, 0 or 1.

Now consider any two points of $A(n;\, 0, \cdots, 0)$. These may be written $x_{n+1}^{i_1} \cdots x_{n+p}^{i_p}$ and $x_{n+1}^{j_1} \cdots x_{n+p}^{j_p}$, where the exponents i_k and j_k are either 0 or 1. The distance between these points can be estimated as follows:

$$\begin{aligned}(x_{n+1}^{i_1} \cdots x_{n+p}^{i_p},\, x_{n+1}^{j_1} \cdots x_{n+p}^{j_p}) &= (e,\, x_{n+p}^{i_p} \cdots x_{n+1}^{-i_1} x_{n+1}^{j_1} \cdots x_{n+p}^{j_p})\\ &\le 2[(e,\, x_{n+1}) + (e,\, x_{n+2} + \cdots + (e,\, x_{n+p})]\\ &\le \frac{2}{9}(e,\, x_n) + \frac{2}{9^2}(e,\, x_n) + \cdots + \frac{2}{9^p}(e,\, x_n)\\ &< (e,\, x_n)/4.\end{aligned}$$

This being true for any two points of $A(n;\, 0, \cdots, 0)$, it follows that $d_n < (e,\, x_n)/4$. On the other hand, e and x_{n+1} are both contained in $A(n;\, 0, \cdots, 0)$, and so $(e,\, x_{n+1}) < d_n$. Consequently,

$$d_{n+1} \le (e,\, x_{n+l})/4 < d_n / 4,$$

and so $d_{n+1} \le d_n / 4$, $n = 1, 2, \cdots$. It follows, in particular, that (iv) is satisfied.

Next, consider any two distinct sets $A(n;\, i_1, \cdots, i_n)$ and $A(n;\, j_1, \cdots, j_n)$ and any two representative points of these sets. These may be written as $a_{i_1 \cdots i_p}$ and $a_{j_1 \cdots j_p}$, where $p \ge n$. Suppose their indices differ first in the k-th place. Then $k \le n$ and we may assume $i_k = 0$ and $j_k = 1$. Let $u = x_{k+1}^{i_{k+1}} \cdots x_p^{i_p}$ and $v = x_k x_{k+1}^{j_{k+1}} \cdots x_p^{j_p}$. Then $(a_{i_1 \cdots i_p},\ a_{j_1 \cdots j_p}) = (u,\, v)$ and

$$(e,\, x_k) \le (e,\, u) + (u,\, v) + (v,\, x_k),$$

$$\begin{aligned}&(e,\, x_{k+1}^{i_{k+1}} \cdots x_p^{i_p}) + (x_{k+1}^{i_{k+1}} \cdots x_p^{i_p},\, x_k x_{k+1}^{j_{k+1}} \cdots x_p^{j_p}) + (x_k x_{k+1}^{j_{k+1}} \cdots x_p^{j_p},\, x_k)\\ &\qquad \le (u,\, v) + 2[(e,\, x_{k+1}) + (e,\, x_{k+2}) + \cdots + (e,\, x_p)]\end{aligned}$$

$$\leq (u, v) + 2(e, x_k)[\frac{1}{9} + \frac{1}{9^2} + \cdots + \frac{1}{9^{n-k}}]$$

$$\leq (u, v) + (e, x_k)/4.$$

Therefore, $(u, v) \geq 3(e, x_k)/4 > 3(e, x_n)/4 \geq 3d_n$. Consequently, any two sets $C(n; i_l, \cdots, i_n)$ of rank n are separated by a distance at least equal to $3d_n$, and condition (v) is satisfied.

Finally, $C(n; i_1, \cdots, i_n)$ is totally bounded, since it is the union of a finite number 2^{p-n} of sets $C(n; i_1, \cdots, i_p)$ having arbitrarily small diameter d_p. Any totally bounded closed subset of a complete space is compact, and therefore (i) is satisfied, provided G is complete with respect to the metric (x, y). If this is not the case, G can be embedded isometrically in a complete space $\overline{G}$ and the same reasoning then shows that the closure of $C(n; i_1, \cdots, i_n)$ in G is compact. Consequently, the sets $C(n; i_1, \cdots, i_n)$ will be compact, if they are all closed in $\overline{G}$. To ensure this, the sequence $\{x_n : n \in N\}$, on which the construction rests, must be subjected to a further condition. Introduce a topologically equivalent metric $\rho(x, y)$ with respect to which G is complete, and let the sequence $\{x_n : n \in N\}$ be so chosen that, it satisfies not only condition (2.1) but also the condition

$$\rho(a_{i_1 \cdots i_n}, a_{i_1 \cdots i_n} x_{n+1}) \leq 2^{-n} (i_1, \cdots, i_n = 0, 1). \tag{2.2}$$

Both conditions are satisfied by any sequence that converges to e sufficiently rapidly. Condition (2.2) implies that for any sequence of subscripts $i_i, i_2, \cdots$ the sequence $a_{i_1 \cdots i_n} (n = 1, 2, \cdots)$ is a Cauchy sequence with respect to the metric ρ, and therefore converges to an element of G. But any point of accumulation of C in G is the limit of such a sequence. Hence C is closed in V, and likewise the sets $C(n; i_1, \cdots, i_n)$. Thus in all cases, we have constructed in G a family of sets having properties (i) to (v).

Now, consider any family of sets $C(n;\, i_1,\, \cdots,\, i_n)$ satisfying conditions (i) to (v). We proceed to define a measure such that $m(C) = 1$. From condition (v), it follows that there exist positive numbers $\{\epsilon_n : n \in N\}$, $\lim_{r\to\infty} \epsilon_r = 0$, such that any two sets $C(n;\, i_1,\, \cdots,\, i_n)$ of rank n are separated by a distance at least equal to $d_n + 2\epsilon_n$. (In the family constructed above, it suffices to take ϵ_n equal to d_n.) Consider the measuring system with $\{U_n : n \in N\}$ equal to the ϵ_n-neighbourhood of $C(n;\, 0,\, \cdots,\, 0)$, and $W_n = 2^{-n}$. Then $\operatorname{diam}(U_n) < d_n + 2\epsilon_n$, hence $\operatorname{diam}(U_n) \to 0$, and any set congruent to U_n can overlap at most one set $C(n;\, i_i,\, \cdots,\, i_n)$ of rank n.

To show that $m(C) = 1$, consider any $r > 0$ and choose n such that $\operatorname{diam}(U_n) < r$. From (iii), it follows that C can be covered by 2^n sets congruent to U_n. Hence $L_r(C) \le 2^n W_n = 1$. To establish the opposite inequality, consider any covering of C by sets of the form $b_i U_{n_i} a_i$. Since C is compact, only finite coverings need be considered. Suppose $C \subseteq \bigcap_{s=1}^{k} b_s U_{n_s} a_s$. Let p be the largest of the indices $n_1,\, n_2,\, \cdots,\, n_k$ and let q_i be the number of sets $C(p;\, i_1,\, \cdots,\, i_p)$ of rank p overlapped by $b_i U_{n_i} a_i$. Since the sets $b_i U_{n_i} a_i$ cover C, we must have $q_1 + q_2 + \cdots + q_k \ge 2^p$. On the other hand, each set $b_i U_{n_i} a_i$ can overlap at most one set of rank n_i, and therefore at most 2^{p-n_i} sets of rank p. Hence $2^{p-n_i} \ge q_i$. Therefore $W_{n_i} = 2^{-n_i} = 2^{p-n_i} / 2^p \ge q_i / 2^p$, and so $\sum_{i=1}^{k} W_{n_i} \ge 2^{-p} \sum_{i=1}^{k} q_i \ge 1$, which shows that $L_r(C) \ge 1$. Thus $L_r(C) = 1$ for every positive r. Therefore, $m(C) = 1$ and we have established the existence of a two-sided-invariant Borel measure in G. This completes the proof of Lemma 2.3. □

Lemma 2.4 (Oxtoby)**.** *In any complete separable metric group with two-sided invariant metric which is dense in itself, there exists a two-sided-invariant quasi-finite Borel measure and no any two-sided-invariant quasi-finite Borel measure possesses a uniqueness property.*

Proof. The structure of the measure m (constructed in Lemma 2.3) within the unit set C is easy to recognize. To each point $x \in C$ corresponds uniquely a sequence of indices $i_i, i_2, \cdots$ such that $x = \bigcap_n C$ $(n; i_i, \cdots, i_n)$. If we regard this sequence $\{i_n : n \in N\}$ as an element of the infinite product group C^* of the group of order two with itself, with the usual topology [21](see p. 51), it is not hard to see that this correspondence is a homeomorphism between C and C^*, and that the measure m in C corresponds to the normalized Haar measure in C^*. Hence, the measure m in G may be regarded as an extension of the measure in C defined by this correspondence and the Haar measure in C^*.

The foregoing construction evidently contains a large degree of arbitrariness. In fact, it associates an invariant measure with any sequence $\{x_n : n \in N\}$ that satisfies conditions (2.1) and (2.2). If a sequence satisfies these conditions, any subsequence will also satisfy them. In particular, the subsequence $\{x_{2n} : n \in N\}$ defines a unit set $C_1 \subseteq C$, such that $m(C_1) = 0$, where m is the measure associated with the sequence $\{x_n : n \in N\}$. Repeated application of this result gives rise to a sequence of measures $m_1, m_2, \cdots$, where m_k is the measure associated with the subsequence $\{x_{k(n)} : k(n) = n2^k\}$. These measures are all distinct and form an increasing sequence, $m_1(A) \leq m_2(A) \leq \cdots$, for every Borel set A. It therefore appears that infinitely many invariant measures always exist, and there is certainly no uniqueness theorem for the problem of measure we have considered. □

3. Main Results

The following two assertions answers to Problems 1.1-1.2 for generators of left (or right)-shy sets in a Polish groups, which is dense in itself.

Theorem 3.1. *In any Polish group G which is dense in itself, there exists a left (or right)-invariant quasi-finite generator of left (or right)-shy sets.*

Proof. By Lemma 2.1, there exists a left-invariant quasi-finite Borel measure μ in G. By Lemma 2.2, the measure μ is a generator of left-shy sets in G. □

By Remark 2.1, we get the following:

Theorem 3.2. *In any Polish group G which is dense in itself, there exists no a generator of left (or right)-shy sets with the property of uniqueness in the class of quasi-finite generators of left (or right)-shy sets.*

Theorems 3.1-3.2 extend the following early obtained facts:

Fact 3.1 ([17], Theorem 2.1, p. 5). Let $(G, +)$ be an abelian Polish group, which admits the following representation $G = G_0 + G_1$, where G_0 is an uncountable locally compact Hausdorff topological group and G_1 is such a group that $G_0 \cap G_1 = \{0\}$, where $\{0\}$ denotes the zero of the group $(G, +)$. Then, there exists a semi-finite inner regular invariant generator λ of left shy sets in G. The generator λ is non-σ-finite iff the G_1 is uncountable.

Fact 3.2 ([17], Theorem 2.2, p. 8). Let G be a non-locally compact abelian Polish group with an invariant metric, which contains any uncountable locally compact Hausdorff topological group. Then, the class of generators of shy sets in G is nonempty and each its element is non-σ-finite.

Fact 3.3 ([17], Corollary 2.1. p. 8)**.** There always exists a semi-finite inner regular generator of shy sets in the Polish topological vector space G and no any generator has a property of uniqueness, if $\dim(G) > 1$. In addition, if G is infinite-dimensional, then every generator of shy sets in G is non-σ-finite.

Remark 3.1. Let G be a Polish group. Then, there does not exist a family of mutually singular generators of left (or right) shy sets in G. This is a simple consequence of Theorem 5.4 established in [18] (see p. 27).

Remark 3.2. Let G be a non-locally compact Polish group. Then each generator m of left-shy sets is non-σ-finite. Indeed, following Kakutani [7], any topological group that satisfies the first countability axiom is metrizable, and that the metric can be taken to be left-invariant. Following Dougherty [4], any compact set in non-locally compact Polish group equipped with left invariant metric, is left-shy set. If we assume that m is σ-finite, then by using a certain result of Oxtoby (see [11], Lemma 1, p. 216), there will be a countable family of compact sets $\{F_k : k \in N\}$ such that $m(G \setminus \cup_{k \in N} F_k) = 0$. Following [9] (see Theorem 3, p. 32), $\cup_{k \in N} F_k$ is left-shy set. The previous relation implies that $G \setminus \cup_{k \in N} F_k$ is left-shy since m is generator of left-shy sets in G. Finally, we get that G is left-shy set as a union of two shy sets $\cup_{k \in N} F_k$ and $G \setminus \cup_{k \in N} F_k$. This is a required contradiction.

The next two assertions answers to Problems 1.1-1.2 for generators of two-sided-shy sets in a Polish groups, which is dense in itself

Theorem 3.3. *In any Polish group G which is dense in itself and is equipped with a two-sided invariant metric, there exists a two-sided invariant quasi-finite generator of two-sided-shy sets.*

Proof. By Lemma 2.3, there exists a two sided invariant quasi-finite Borel measure μ in G. By Lemma 2.2, the measure μ is a generator of two-sided-shy sets in G. □

By Lemma 2.4, we get the following:

Theorem 3.4. *In any Polish group G which is dense in itself and is equipped with a two-sided invariant metric, there exists no a generator of two-sided-shy sets with the property of uniqueness in the class of quasi-finite generators of two-sided-shy sets.*

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